

ABSOLUTE AND UNCONDITIONAL CONVERGENCE
IN BANACH SPACES

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ABSOLUTE AND UNCONDITIONAL CONVERGENCE IN BANACH SPACES

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Let us denote by I the class of real-valued functions defined on the closed interval $[0, 1]$ which are indefinite Lebesgue integrals, by AC the class of real-valued functions which are absolutely continuous in $[0, 1]$, by BV the class of real-valued functions of bounded variation in $[0, 1]$, and by D the class of real-valued functions which are differentiable almost everywhere in $[0, 1]$. Then the following theorem is well known and contains what is usually referred to as the Fundamental Theorem of Integral Calculus.

THEOREM 0.1.

- (a) $I = AC \subset BV \subset D$.
- (b) *If an integral is almost everywhere differentiable, its derivative is almost everywhere equal to the integrand.*

If, instead of real-valued functions, we consider functions whose ranges lie in a fixed but arbitrary Banach space, part (b) of Theorem 0.1 holds for several well-known definitions of integration (see, for example, [9; 300–301]), but part (a) breaks down. Relations of the type stated in part (a) may hold, however, if the class of range spaces under consideration is suitably restricted. For example, Clarkson [5] has proved that a sufficient (but not necessary) condition that $BV \subset D$ is that the range space be uniformly convex. We shall investigate in this paper the nature of range spaces for which other such relations hold.

In particular, we shall consider the relations $AC \subset BV$ and $I = AC$, where AC and BV are defined in terms of the strong topology and I is the class of integrals defined by Bochner and Dunford (see [4], [6]). These questions lead to a consideration of the relation between absolute and unconditional convergence of series which in turn involves the discussion of a generalized notion of orthogonality. Sections 2 and 3 are concerned with convergence and orthogonality respectively.

The author stated in an abstract (Bulletin of the American Mathematical Society; abstract 48–7–241) that finite dimensionality of a Banach space was necessary and sufficient for the equivalence of unconditional and absolute convergence of series in the space. An error in his proof was discovered, and the paper was not published in full. The truth or falsity of this theorem remains an open question. Theorem 2.2 of the present paper throws some light on the matter, but fails to clear it up completely.

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1. **Relations between the classes I , AC , and BV .** Let E_0 be a fixed elementary figure in Euclidean n -space, R_n , and let $\Phi(E)$ be an additive function defined for all elementary figures $E \subset E_0$ and assuming values in a Banach space. Then we say that $\Phi(E)$ is of *bounded variation* on E_0 if and only if there exists a number M such that for every finite set $\{E_i\}$ of disjoint elementary figures in E_0

$$\sum_{i=1}^m ||\Phi(E_i)|| \leq M.$$

We say that $\Phi(E)$ is *absolutely continuous* on E_0 if and only if

$$\lim_{|E| \rightarrow 0} ||\Phi(E)|| = 0.$$

THEOREM 1.1. *If B is a Banach space in which unconditional convergence of series does not imply absolute convergence, then there exists a function $\Phi(I)$ defined for all subintervals I of the unit interval $[0, 1]$ and assuming values in B such that $\Phi(I)$ is additive and absolutely continuous on $[0, 1]$ but is not of bounded variation on $[0, 1]$.*

Let $\sum_n \xi_n$ be an unconditionally but not absolutely convergent series of elements of B . First we define the point functions

$$\phi(x) = \begin{cases} 2^n \xi_n & \text{for } x \in I_n = E \left\{ \frac{1}{2^n} \leq x < \frac{1}{2^{n-1}} \right\} \\ \theta & \text{for } x = 0, 1, \end{cases} \quad (n = 1, 2, 3, \dots)$$

where θ is the zero element of B ;

$$\phi_i(x) = \begin{cases} \phi(x) & \text{for } \frac{1}{2^i} \leq x < 1 \\ \theta & \text{for } 0 \leq x < \frac{1}{2^i} \end{cases} \quad (i = 1, 2, 3, \dots).$$

Then obviously

$$\lim_{i \rightarrow \infty} \phi_i(x) = \phi(x) \quad (x \in [0, 1]).$$

If E is any measurable subset of $[0, 1]$, each $\phi_i(x)$ is integrable on E in the sense of Pettis with

$$\int_E \phi_i(x) dx = \sum_{n=1}^i M_n^E \xi_n,$$

where $M_n^E = 2^n |E \cdot I_n|$ ($n = 1, 2, 3, \dots$). We obviously have $0 \leq M_n^E \leq 1$ for each E and each n ; therefore, since $\sum_n \xi_n$ is unconditionally convergent, it follows ([3; Theorem 8]) that

$$\lim_{i \rightarrow \infty} \int_E \phi_i(x) dx = \lim_{i \rightarrow \infty} \sum_{n=1}^i M_n^E \xi_n = \sum_{n=1}^{\infty} M_n^E \xi_n$$

exists for every measurable $E \subset [0, 1]$. Hence $\phi(x)$ is integrable in the sense of Pettis [9; Theorem 4.1] and

$$\int_E \phi(x) dx = \sum_{n=1}^{\infty} M_n^E \xi_n.$$

Therefore, the function

$$\Phi(I) = \int_I \phi(x) dx$$

is defined for every subinterval I of $[0, 1]$, assumes values in B , and is additive and absolutely continuous ([9; Theorems 2.4 and 2.5]) in $[0, 1]$.

Now for each j ,

$$M_n^{I_j} = \begin{cases} 0 & \text{for } n \neq j, \\ 1 & \text{for } n = j, \end{cases} \quad \Phi(I_j) = \xi_j,$$

and therefore,

$$\sum_{j=1}^{\infty} \|\Phi(I_j)\| = \sum_{j=1}^{\infty} \|\xi_j\| = \infty;$$

hence given any number M , there exists a finite set of integers $j_1, j_2, \dots, j_k$ such that

$$\sum_{i=1}^k \|\Phi(I_{j_i})\| > M,$$

and so $\Phi(I)$ is not of bounded variation on $[0, 1]$.

Two remarks should be made about the statement of Theorem 1.1. First, we note that the function Φ is defined not only for every subinterval of $[0, 1]$ but for every measurable subset of $[0, 1]$ and that (under proper extensions of our definitions of these terms) it retains on this extended domain the properties of additivity and absolute continuity, yet still is not of bounded variation. Secondly, it is easily seen from the methods employed that such a function can be constructed on the measurable subsets of any bounded measurable set (of non-zero measure) in R_n .

THEOREM 1.2. *If B is a Banach space in which unconditional convergence of series implies absolute convergence and if E_0 is any fixed elementary figure in R_n , then every additive and absolutely continuous function defined for all elementary figures $E \subset E_0$ and assuming values in B is of bounded variation on E_0 .*

Suppose $\Phi(E)$ is additive and absolutely continuous on E_0 but not of bounded variation on E_0 . Let $E_1^1, E_2^1, \dots, E_{k_1}^1$ be a subdivision of E_0 into elementary figures such that $\|E_i^1\| < 1$ ($i = 1, 2, \dots, k_1$) and also such that

$$\sum_{i=1}^{k_1} \|\Phi(E_i^1)\| > 1.$$

On at least one of the elementary figures E_i^1 , $\Phi(E)$ is not of bounded variation. We shall denote one such elementary figure by E^1 . Now for each n ($n = 2, 3, 4, \dots$) we let $E_1^n, E_2^n, \dots, E_{k_n}^n$ be a subdivision of E^{n-1} (E^{n-1} being defined for each n as it is above for $n = 2$) such that $|E_i^n| < 1/n$ ($i = 1, 2, \dots, k_n$) and also such that

$$\sum_{i=1}^{k_n} \|\Phi(E_i^n)\| > 1.$$

Now we denote by $\{\xi_i\}$ the sequence

$$\Phi(E_1^1), \dots, \Phi(E_{k_1}^1), \Phi(E_1^2), \dots, \Phi(E_{k_2}^2), \Phi(E_1^3), \dots, \Phi(E_{k_3}^3), \dots$$

(the elements $\Phi(E^n)$ being omitted).

Given $\epsilon > 0$, it follows from the absolute continuity of $\Phi(E)$ that there exists $\delta = \delta(\epsilon)$ such that if $|E| < \delta$, $\|\Phi(E)\| < \epsilon$. Furthermore, if $N > 1/\delta$, $|E^N| < \delta$. Clearly there exists a number J such that if $J < j_1 < j_2 < \dots < j_m$, the elements ξ_{j_ν} ($\nu = 1, 2, \dots, m$) are of the form $\Phi(E_\nu)$ where $E_\nu \subset E^N$ ($\nu = 1, 2, \dots, m$). We let $E^* = \sum_{\nu=1}^m E_\nu$; then noting that $|E^*| \leq |E^N| < \delta$ and using the additivity of $\Phi(E)$, we have

$$\left\| \sum_{\nu=1}^m \xi_{j_\nu} \right\| = \left\| \sum_{\nu=1}^m \Phi(E_\nu) \right\| = \left\| \Phi(E^*) \right\| < \epsilon.$$

This proves the unconditional convergence of $\sum_i \xi_i$.

It follows from the definition of $\{\xi_i\}$ that

$$\sum_{j=1}^{\infty} \left\| \xi_j \right\| = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{k_n} \left\| \Phi(E_i^n) \right\| - \left\| \Phi(E^n) \right\| \right).$$

Now $\sum_{i=1}^{k_n} \|\Phi(E_i^n)\| > 1$, and because $\Phi(E)$ is absolutely continuous $\lim_{n \rightarrow \infty} \|\Phi(E^n)\| = 0$. Thus

$$\sum_{j=1}^{\infty} \|\xi_j\| \geq \lim_{m \rightarrow \infty} m[1 + o(1)] = \infty.$$

The series $\sum_i \xi_i$ thus demonstrates a contradiction of our hypotheses, and the theorem is proved.

In order to summarize these results in the light of the remarks made in the introduction, we introduce the following notation. If E_0 is a fixed elementary figure in R_n and if B is a Banach space, we shall denote by $I_{BD}(E_0, B)$ the class of all Bochner-Dunford indefinite integrals defined for elementary figures in E and assuming values in B , by $I_P(E_0, B)$ the class of all Pettis indefinite integrals defined from E_0 to B , by $AC(E_0, B)$ the class of all additive absolutely continuous functions of elementary figures defined from E_0 to B ; and by $BV(E_0, B)$ the class of all additive functions of elementary figures defined from E_0 to B which are of bounded variation on E_0 .

THEOREM 1.3. *If E_0 is a fixed but arbitrary elementary figure in R_n and if B*

is a Banach space, equivalence of unconditional and absolute convergence of series in B is necessary and sufficient that

- (a) $AC(E_0, B) \subset BV(E_0, B)$,
- (b) $I_P(E_0, B) \subset BV(E_0, B)$.

Part (a) is merely a restatement of Theorems 1.1 and 1.2. The proof of sufficiency in part (b) consists of noting that $I_P(E_0, B) \subset AC(E_0, B)$ for all B . Necessity follows from the fact that the example constructed in the proof of Theorem 1.1 is a Pettis integral.

THEOREM 1.4. *If E_0 is a fixed but arbitrary elementary figure in R_n and if B is a Banach space such that every additive absolutely continuous function of elementary figures defined from E_0 to B is strongly differentiable almost everywhere in E_0 , then equivalence of unconditional and absolute convergence of series in B is necessary and sufficient that*

- (a) $I_{BD}(E_0, B) = AC(E_0, B)$,
- (b) $I_{BD}(E_0, B) = I_P(E_0, B)$.

Necessity of the condition follows immediately from Theorem 1.3 and the fact that the Bochner-Dunford indefinite integral is a function of bounded variation. This property of the Bochner-Dunford integral seems to be well known (see, for example [10]); however we have been unable to find an explicit proof of the property. It follows immediately from the absolute convergence of the Bochner-Dunford integral and the bounded variation of the Lebesgue integral. The differentiability of additive absolutely continuous functions guarantees ([9; 300]) that relations (a) and (b) hold provided $AC(E_0, B) \subset BV(E_0, B)$; thus sufficiency of the condition follows immediately from Theorem 1.3.

The differentiability hypothesis in Theorem 1.4 is unnecessarily strong. In §2 we shall replace it by a weaker restriction, but we have been unable as yet to remove the qualifying clause entirely.

2. Convergence of series. In the study of unconditional and absolute convergence of series it is convenient to use the notion of a "maximal" sequence. Roughly speaking, this is a sequence for which the norm of a partial sum is not increased by the removal of terms from the sum. This leads to a series for which a certain proof of simple convergence automatically guarantees unconditional convergence. The notion of "maximal" sequence is defined precisely as follows:

DEFINITION 2.1. A Banach space is said to have the *maximal sequence property* (m.s.p.) under the following conditions:

If the space contains any infinite sequence of linearly independent elements, it contains one such sequence $\{\eta_k\}$ such that $\|\eta_k\| = 1$ ($k = 1, 2, 3, \dots$) and

$$\left\| \sum_{k \in K'_n} c_k \eta_k \right\| \leq \left\| \sum_{k \in K_n} c_k \eta_k \right\| \quad (K'_n \subset K_n; n = 1, 2, 3, \dots).$$

Here K_n is the set of integers k such that

$$\frac{1}{2}n(n-1) + 1 \leq k \leq \frac{1}{2}n(n+1)$$

and $c_k = \pm 1$ for $k \in K_n$.

The sequence $\{\eta_k\}$ is said to be a *maximal* sequence.

It should be noted that all finite dimensional Banach spaces have m.s.p. Such spaces satisfy the definition vacuously.

THEOREM 2.2. *If B is a Banach space having m.s.p., a necessary and sufficient condition that unconditional and absolute convergence of series be equivalent in B is that B be finite dimensional.*

Absolute convergence always implies unconditional convergence, and in finite dimensional spaces it is easily seen that unconditional convergence implies absolute convergence. This implication holds for series of real numbers, hence for each component of our series separately. Since there are only a finite number of components, the desired result follows immediately.

It remains to prove that if B is infinite dimensional and has m.s.p., then B contains a series which is unconditionally but not absolutely convergent. Let $\{\eta_k\}$ ($k = 1, 2, 3, \dots$) be a maximal sequence in B . We shall construct a countable set $\{\xi_i\}$ of linear combinations of the η_k such that $\sum \xi_i$ is unconditionally but not absolutely convergent. The nature of this construction will depend on the behavior of the expression $\|\sum_k c_k \eta_k\|$ as a function of the real coefficients c_k . We define two types of infinite matrices $\{a_{ik}\}$ and $\{b_{ik}\}$ ($i, k = 1, 2, 3, \dots$) and show that one of the two types of transformations

$$\xi_i = \sum_k a_{ik} \eta_k, \quad \xi_i = \sum_k b_{ik} \eta_k$$

will (when multiplied by suitable constants) give the required set $\{\xi_i\}$.

The matrix $\{a_{ik}\}$. For each integer $n > 0$, let P_n be a rectangular matrix of real numbers having 2^n rows and n columns; let the absolute value of each element of P_n be unity, and let the signs of the elements be so chosen that no two rows are identical. In other words, for the rows of P_n we take all possible ordered sets of n elements of the form ± 1 . The matrix $\{a_{ik}\}$ is formed by placing the P_n corner to corner in a diagonal array and filling in the rest of the doubly infinite array with zeros.

The matrix $\{b_{ik}\}$. The matrix $\{b_{ik}\}$ is defined as follows:

$$|b_{ik}| = \begin{cases} 1/i & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases}$$

We introduce the following notation. In the matrix $\{a_{ik}\}$ let

- (a) K_n denote the set of column indices corresponding to P_n ,
- (b) I_n denote the set of row indices corresponding to P_n ,
- (c) $|K'_n|$ (read measure of K'_n) denote the number of integers contained in a set $K'_n \subset K_n$. For $I'_n \subset I_n$, $|I'_n|$ will have a similar meaning.

We now define the following functions:

$$\mathfrak{N}(K'_n, i) = \left\| \sum_{k \in K'_n} a_{ik} \eta_k \right\| \quad (K'_n \subset K_n; i \in I_n; n = 1, 2, 3, \dots),$$

$$\mathfrak{M}(K'_n, q) = \left\{ \sum_{k \in K'_n} |a_{ik}|^q \right\}^{1/q} \quad (q \geq 1; K'_n \subset K_n; i \in I_n; n = 1, 2, 3, \dots).$$

The function $\mathfrak{M}$ obviously does not depend on i . Clearly $\mathfrak{N}$ depends on the sequence $\{\eta_k\}$, but no confusion will arise from the fact that our notation does not indicate this dependence.

The construction of the sequence $\{\xi_i\}$ will be discussed in two cases.

Case I. There exist a number q with $1 \leq q < 2$ and an infinite sequence $\{n_\nu\}$ of integers such that

$$\mathfrak{N}(K_{n_\nu}, i) \geq \mathfrak{M}(K_{n_\nu}, q) \quad (i \in I_{n_\nu}; \nu = 1, 2, 3, \dots).$$

The numbers n_ν will appear throughout Case I as subscripts. To simplify the notation, we shall use the single subscript ν in place of the double subscript n_ν . Thus K_ν means K_{n_ν} , etc.

We now define a matrix $\{a'_{ik}\}$ as follows:

$$a'_{ik} = \begin{cases} a_{ik}/n_\nu 2^{n_\nu} & \text{for } i \in I_\nu; k \in K_\nu; \nu = 1, 2, 3, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Then we set

$$\xi_i = \sum_{k=1}^{\infty} a'_{ik} \eta_k$$

which clearly reduces to a finite sum. Thus we have that for each ν ($\nu = 1, 2, 3, \dots$)

$$\begin{aligned} D_\nu &= \sum_{i \in I_\nu} \|\xi_i\| \geq |I_\nu| \cdot \frac{1}{n_\nu 2^{n_\nu}} \cdot \min_{i \in I_\nu} \mathfrak{N}(K_\nu, i) \\ (2.2.1) \quad &\geq \frac{1}{n_\nu} \mathfrak{M}(K_\nu, q) = 1/n_\nu^{1-1/q}. \end{aligned}$$

For each $I'_\nu \subset I_\nu$ ($\nu = 1, 2, 3, \dots$), we have

$$\begin{aligned} \left\| \sum_{i \in I'_\nu} \xi_i \right\| &= \left\| \sum_{i \in I'_\nu} \sum_{k \in K_\nu} a'_{ik} \eta_k \right\| = \left\| \sum_{k \in K_\nu} \sum_{i \in I'_\nu} a'_{ik} \eta_k \right\| \\ (2.2.2) \quad &\leq \sum_{k \in K_\nu} \left\| \sum_{i \in I'_\nu} a'_{ik} \eta_k \right\| = \sum_{k \in K_\nu} \left| \sum_{i \in I'_\nu} a'_{ik} \right|; \end{aligned}$$

whence for each ν

$$(2.2.3) \quad \max_{I'_\nu \subset I_\nu} \left\| \sum_{i \in I'_\nu} \xi_i \right\| \leq \max_{I'_\nu \subset I_\nu} \sum_{k \in K_\nu} \left| \sum_{i \in I'_\nu} a'_{ik} \right|.$$

Suppose now that I'_ν consists of the indices of all rows of P_ν which contain at least as many positive terms as negative ones. Then the matrix P'_ν , formed with the rows from P_ν designated by the indices from I'_ν , has the property that the sum of the elements of each column is positive. This is shown as follows: Clearly

$$|I'_\nu| = \sum_{s=0}^{[n_\nu/2]} \binom{n_\nu}{s};$$

and if M denotes the maximum number of negative entries in any column of P'_ν , then

$$M = \sum_{s=0}^{[(n_\nu-2)/2]} \binom{n_\nu-1}{s}.$$

We must now consider two cases. First, suppose n_ν is even; then

$$|I'_\nu| = \sum_{s=0}^{n_\nu/2} \binom{n_\nu}{s} > 2^{n_\nu-1},$$

while

$$M = \sum_{s=0}^{(n_\nu-2)/2} \binom{n_\nu-1}{s} = 2^{n_\nu-2}.$$

In case n_ν is odd,

$$|I'_\nu| = \sum_{s=0}^{(n_\nu-1)/2} \binom{n_\nu}{s} = 2^{n_\nu-1},$$

while

$$M = \sum_{s=0}^{(n_\nu-3)/2} \binom{n_\nu-1}{s} < 2^{n_\nu-2}.$$

Thus in either case $M < \frac{1}{2} |I'|$; so the sum of the elements in each column is positive.

Because each of the sums over i is positive, the absolute value signs may be removed from the last member of (2.2.2). The resulting double sum will then have the property that it will not be increased by any variation in I'_ν ; since any inclusion of an additional index will add terms whose total is negative, and similarly, any omission of a present index will remove terms whose total is positive or zero. Thus this particular choice of I'_ν gives a relative maximum for the last member of (2.2.2)—namely, the maximum over all subsets of I_ν such that the sum of the coefficients in each column will be positive. It follows immediately from the symmetry of the matrices P_n that this relative maximum

is equal to the maximum over all subsets of I_ν determined by an arbitrary assignment of algebraic sign to the sum of the coefficients in each column. Thus this choice of I'_ν will give us the desired maximum, namely the right member of (2.2.3); and for this choice of I'_ν we have

$$\begin{aligned} c'_\nu &= \sum_{k \in K_\nu} \left| \sum_{i \in I'_\nu} a'_{ik} \right| = \sum_{k \in K_\nu} \sum_{i \in I'_\nu} a'_{ik} = \sum_{i \in I'_\nu} \sum_{k \in K_\nu} a'_{ik} \\ &= \frac{\sum_{s=0}^{[n_\nu/2]} (n_\nu - 2s) \binom{n_\nu}{s}}{n_\nu 2^{n_\nu}} < \frac{\sum_{s=0}^{n_\nu} |n_\nu - 2s| \binom{n_\nu}{s}}{n_\nu 2^{n_\nu}}. \end{aligned}$$

We now wish to obtain an estimate of this expression in terms of n_ν for each ν . (The estimate of c'_ν given below is due to W. S. Ament.) We fix ν and for simplicity temporarily drop it as a subscript. Then by the aid of Cauchy's inequality and the relation $\sum_{s=0}^n \binom{n}{s} = 2^n$, we obtain

$$\begin{aligned} c'_\nu &< \frac{\sum_{s=0}^n |n - 2s| \binom{n}{s}}{n 2^n} \leq \left\{ \frac{\sum_{s=0}^n (n - 2s)^2 \binom{n}{s}}{n^2 2^n} \right\}^{\frac{1}{2}} \left\{ \frac{\sum_{s=0}^n \binom{n}{s}}{2^n} \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{\sum_{s=0}^n [(n - s)^2 - 2s(n - s) + s^2] \binom{n}{s}}{n^2 2^n} \right\}^{\frac{1}{2}}. \end{aligned}$$

From the symmetry of the set of binomial coefficients we have

$$\sum_{s=0}^n (n - s)^2 \binom{n}{s} = \sum_{s=0}^n s^2 \binom{n}{s},$$

whence

$$\begin{aligned} c'_\nu &< \left\{ \frac{\sum_{s=0}^n s^2 \binom{n}{s} - \sum_{s=0}^n s(n - s) \binom{n}{s}}{n^2 2^{n-1}} \right\}^{\frac{1}{2}} \\ &= \frac{1}{2^{(n-1)/2}} \left\{ \sum_{s=0}^n \frac{s^2}{n^2} \binom{n}{s} - \sum_{s=0}^n \frac{s(n - s)}{n^2} \binom{n}{s} \right\}^{\frac{1}{2}}. \end{aligned}$$

But we have also

$$\begin{aligned} \sum_{s=0}^n \frac{s^2}{n^2} \binom{n}{s} &= \sum_{s=1}^n \frac{s}{n} \binom{n-1}{s-1} = \frac{1}{n} \left\{ \sum_{s=1}^n \binom{n-1}{s-1} + \sum_{s=1}^n (s-1) \binom{n-1}{s-1} \right\} \\ &= \frac{1}{n} \left\{ 2^{n-1} + \left[\frac{d}{dz} (1+z)^{n-1} \right]_{z=1} \right\} = \frac{1}{n} \left\{ 2^{n-1} + (n-1)2^{n-2} \right\} \end{aligned}$$

and

$$\sum_{s=0}^n \frac{s(n-s)}{n^2} \binom{n}{s} = \sum_{s=0}^{n-1} \frac{s}{n} \binom{n-1}{s} = \sum_{s=1}^{n-1} \frac{n-1}{n} \binom{n-2}{s-1} = \frac{n-1}{n} 2^{n-2},$$

whence

$$c'_\nu < \left\{ \frac{2^{n-1}}{n2^{n-1}} + \frac{2^{n-2}}{2^{n-1}} \left[\frac{n-1}{n} - \frac{n-1}{n} \right] \right\}^{\frac{1}{2}} = n^{-\frac{1}{2}}.$$

Thus if $\{\xi_{i_\mu}\}$ is any subsequence of $\{\xi_i\}$, we have

$$c_\nu = \left\| \sum_{i_\mu \in I_\nu} \xi_{i_\mu} \right\| < n_\nu^{-\frac{1}{2}} \quad (\nu = 1, 2, 3, \dots).$$

Comparing this with (2.2.1), we see

$$\lim_{\nu \rightarrow \infty} \frac{D_\nu}{c_\nu} \geq \lim_{\nu \rightarrow \infty} \frac{n_\nu^{\frac{1}{2}}}{n_\nu^{1-1/q}} = \lim_{\nu \rightarrow \infty} n_\nu^{1/q-1/2} = \infty$$

since $q < 2$.

Because of the homogeneity of the expressions involved, it is clear that the ratios D_ν/c_ν remain unchanged if for the matrix $\{a'_{ik}\}$ above we substitute the matrix $\{a''_{ik}\}$ defined as follows:

$$a''_{ik} = \begin{cases} g_\nu a'_{ik} & \text{for } i \in I_\nu; \quad k \in K_\nu; \quad \nu = 1, 2, 3, \dots, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{g_\nu\}$ ($\nu = 1, 2, 3, \dots$) is any sequence of real numbers. Since $D_\nu/c_\nu \rightarrow \infty$, it is possible to choose the numbers g_ν in such a way that if

$$\xi'_i = \sum_{k=1}^{\infty} a'_{ik} \eta_k \quad (i = 1, 2, 3, \dots),$$

$$D'_\nu = \sum_{i \in I_\nu} \|\xi_i\| \quad (\nu = 1, 2, 3, \dots),$$

and

$$c'_\nu = \max_{I'_\nu \subset I_\nu} \left\| \sum_{i \in I'_\nu} \xi_i \right\|$$

then $([8; 302]) \sum_{\nu=1}^{\infty} c'_\nu$ will be convergent while $\sum_{\nu=1}^{\infty} d''_\nu = \infty$. Thus a suitable choice of the sequence $\{g_\nu\}$ yields a series which is unconditionally but not absolutely convergent.

Case II. For each number q ($1 \leq q < 2$) there exists a number $N = N(q)$ such that for each $n > N$ there is an index $i_n \in I_n$ such that

$$\mathfrak{N}(K_n, i_n) < \mathfrak{N}(K_n, q).$$

It should be noted that Cases I and II are exhaustive and mutually exclusive; hence the construction of a series under the hypotheses of Case II will complete the proof of the theorem.

We choose and fix $q > 1$ and assume with no loss of generality that $N(q) = 0$. Let $m = m(n)$ be the smallest integer in the set K_n ; i.e.,

$$m = \frac{1}{2}n(n-1) + 1.$$

Let $\{i_\mu\}$ be any sequence of integers and let K'_n be the set of integers of $\{i_\mu\}$ contained in K_n . Then using the maximal property assumed for $\{\eta_k\}$ and the hypotheses of Case II, we have

$$\begin{aligned} \left\| \sum_{k \in K'_n} \frac{a_{i_n k}}{m} \eta_k \right\| &\leq \left\| \sum_{k \in K_n} \frac{a_{i_n k}}{m} \eta_k \right\| < \left\{ \sum_{k \in K_n} \left| \frac{a_{i_n k}}{m} \right|^q \right\}^{1/q} \\ &= \left\{ n \left(\frac{1}{m} \right)^q \right\}^{1/q} = \frac{n^{1/q}}{m} \quad (n = 1, 2, 3, \dots). \end{aligned}$$

We now study the effects of changing the coefficients $a_{i_n k}$ to the coefficients b_{kk} . For $k \in K'_n$ this involves an estimate of

$$\begin{aligned} \left\| \sum_{k \in K'_n} \frac{a_{i_n k}}{k} \eta_k \right\| &= \left\| \sum_{k \in K'_n} a_{i_n k} \left(\frac{1}{m} - \frac{k-m}{mk} \right) \eta_k \right\| \\ &\leq \left\| \sum_{k \in K'_n} \frac{a_{i_n k}}{m} \eta_k \right\| + \sum_{k \in K'_n} \left\| \frac{(k-m)a_{i_n k}}{mk} \eta_k \right\| \\ &< \frac{n^{1/q}}{m} + \sum_{k \in K'_n} \frac{k-m}{mk} \\ &\leq \frac{n^{1/q}}{m} + |K'_n| \frac{n}{m^2} \leq \frac{n^{1/q}}{m} + \frac{n^2}{m^2} \\ &= O(n^{1/q-2}) + O(n^{-2}) = O(n^{1/q-2}). \end{aligned}$$

Thus we have the result

$$(2.2.4) \quad \left\| \sum_{k \in K'_n} b_{kk} \eta_k \right\| = O(n^{1/q-2}).$$

Now let

$$\xi_i = \sum_{k=1}^{\infty} b_{ik} \eta_k = b_{ii} \eta_i \quad (i = 1, 2, 3, \dots);$$

then

$$\sum_{i=1}^{\infty} \left\| \xi_i \right\| = \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

On the other hand for any subsequence $\{\xi_{i_\mu}\}$ we have

$$\left\| \sum_{\mu=r}^{\infty} \xi_{i_\mu} \right\| \leq \sum_{j=r}^{\infty} \left\| \sum_{i \in K'_j} \xi_i \right\| \quad (i_r \in K_n);$$

but by (2.2.4)

$$\sum_{j=n}^{\infty} \left\| \sum_{i \in K'j} \xi_i \right\| = \sum_{j=n}^{\infty} O(j^{1/q-2}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

since $q > 1$; hence

$$\left\| \sum_{\mu=r}^{\infty} \xi_{i_{\mu}} \right\| \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Thus the series $\sum_i \xi_i$ is unconditionally but not absolutely convergent.

In the light of Theorem 2.2 the results of §1 can be restated as follows:

THEOREM 2.3. *If E_0 is a fixed but arbitrary elementary figure in R_n and if B is a Banach space having m.s.p., each of the following statements is true if and only if B is finite dimensional:*

- (a) $AC(E_0, B) \subset BV(E_0, B)$,
- (b) $I_{BD}(E_0, B) = AC(E_0, B)$,
- (c) $I_P(E_0, B) \subset BV(E_0, B)$,
- (d) $I_P(E_0, B) = I_{BD}(E_0, B)$.

For statements (a) and (c) this is merely a restatement of Theorem 1.3 in the light of Theorem 2.2. For statements (b) and (d) we note that in Theorem 1.4 the differentiability hypothesis is used only in the proof of sufficiency. Now finite dimensionability implies uniform convexity which in turn implies ([5; Theorem 6]) the differentiability hypothesis of Theorem 1.3.

3. Orthogonal vectors. The fact that in Hilbert space an orthogonal sequence is a maximal sequence suggests that orthogonality might be a useful tool in the study of m.s.p. With this in mind we suggest the following generalization of orthogonality. (For other generalizations of orthogonality see [7].)

DEFINITION 3.1. If α and β are elements of a Banach space α is orthogonal to β ($\alpha \perp \beta$) if and only if

$$\|\alpha + \beta\| = \|\alpha - \beta\|.$$

That this definition is consistent with the usual notion of orthogonality may be seen from

THEOREM 3.2. *In a Banach space in which a scalar product is defined $\alpha \perp \beta$ if and only if $\alpha \cdot \beta = 0$.*

This follows immediately from the fact that

$$\|\alpha + \beta\|^2 = (\alpha + \beta) \cdot (\alpha + \beta) = \alpha \cdot \alpha + 2\alpha \cdot \beta + \beta \cdot \beta,$$

$$\|\alpha - \beta\|^2 = (\alpha - \beta) \cdot (\alpha - \beta) = \alpha \cdot \alpha - 2\alpha \cdot \beta + \beta \cdot \beta.$$

The following properties of orthogonal vectors should be noted.

3.3. *If $\alpha \perp \beta$, then $\beta \perp \alpha$ and $\alpha \perp -\beta$.*

3.4. *If $\alpha \perp \beta$, then $\|\alpha\| \leq \|\alpha + \beta\|$.*

Property 3.3 is obvious. To prove 3.4, suppose that $\alpha \perp \beta$ and $\|\alpha\| > \|\alpha + \beta\|$. Then

$$2\|\alpha\| = \|\alpha + \beta + \alpha - \beta\| \leq \|\alpha + \beta\| + \|\alpha - \beta\| < 2\|\alpha\|.$$

This contradiction establishes 3.4.

DEFINITION 3.5. A Banach space B is said to have *orthogonalization property I* (o.p. I) provided that to each integer $n \geq 2$ there corresponds an integer m such that if B' is any m -dimensional subspace of B , then B' contains a set $\{\eta_i\}$ ($i = 1, 2, \dots, n$) such that $\|\eta_i\| = 1$ and

$$\eta_i \perp \sum_{j=1}^n c_j \eta_j \quad (i = 1, 2, \dots, n)$$

for every set $\{c_j\}$ of coefficients which assume values ± 1 or 0 with the provision that $c_i = 0$.

DEFINITION 3.6. If in Definition 3.5 the further restriction $c_j = 0$ for $j \geq i$ is imposed, the property described will be called *orthogonalization property II* (o.p. II).

As in the case of m.s.p. we shall say that all finite dimensional Banach spaces have o.p. I and o.p. II. Such spaces satisfy Definitions 3.5 and 3.6 vacuously if m is chosen sufficiently large in all cases.

THEOREM 3.7. O.p. I implies m.s.p.

It is clear from the definition of o.p. I that the η_k can be chosen so that for each n ($n = 1, 2, 3, \dots$)

$$\eta_j \perp \sum_{k \in K'_n} c_k \eta_k \quad (j \in K_n; K'_n \subset K_n),$$

where $c_k = \pm 1$ for $k \in K_n$ except that $c_j = 0$. The maximal property then follows from 3.4. Linear independence of the η_k may be guaranteed by choosing the different finite sets from linearly independent subspaces.

It appears from the following that 3.1 is a comparatively weak definition of orthogonality. If $\alpha \perp \beta$, it is not necessarily true in general that $\alpha \perp n\beta$. Furthermore, if $\alpha \perp \beta$ and $\alpha \perp \gamma$, it is not true in general that $\alpha \perp \beta + \gamma$. Consider the space $l_\infty^{(4)}$ of all 4-tuples $\xi = (x_1, x_2, x_3, x_4)$ with $\|\xi\| = \max_i |x_i|$. Let $\alpha = (\frac{1}{2}, \frac{3}{4}, 0, 0)$, $\beta = (1, -\frac{3}{4}, 0, 0)$; then $\|\alpha\| = \|\beta\| = 1$ and $\alpha \perp \beta$, but it is not true that $\alpha \perp 2\beta$. Now let $\alpha = (\frac{1}{2}, 1, 0, 0)$, $\beta = (\frac{1}{2}, 0, 1, 0)$, $\gamma = (\frac{1}{2}, 0, 0, 1)$; then $\|\alpha\| = \|\beta\| = \|\gamma\| = \|\beta + \gamma\| = 1$, $\alpha \perp \beta$ and $\alpha \perp \gamma$ but it is not true that $\alpha \perp \beta + \gamma$.

We make these remarks on the weakness of Definition 3.1 in order to suggest that the hypothesis of m.s.p. in Theorem 2.2 may not be so restrictive as it appears to be. In fact, we suggest the following.

CONJECTURE 3.8. Every Banach space has o.p. I hence m.s.p.

We have been unable to prove 3.8, but it is interesting to note that the construction of the required sequence of orthogonal vectors is trivial in every well-known Banach space. The space M serves as an easy example. Let α_n correspond to the function $\alpha_n(t)$ defined as follows:

$$\alpha_n(t) = \begin{cases} 1 & \text{for } \frac{1}{2^n} \leq t \leq \frac{1}{2^{n-1}}, \\ 0 & \text{for } 0 \leq t < \frac{1}{2^n} \text{ and } \frac{1}{2^{n-1}} < t \leq 1. \end{cases}$$

Then clearly $\|\alpha_n\| = 1$ ($n = 1, 2, 3, \dots$) and furthermore for each n ($n = 1, 2, 3, \dots$), $\alpha_n \perp \sum_{i \neq n} \alpha_i$ — a condition much stronger than that guaranteed by o.p. I.

One possible approach to the proof of 3.8 is suggested by the following theorem.

THEOREM 3.9. *Every Banach space has o.p. II.*

To prove this we shall show that given an integer $n \geq 2$, every Banach space of dimension $m + 1$ contains a set $\{\alpha_i\}$ of n elements such that $\|\alpha_i\| = 1$ and

$$\alpha_i \perp \sum_{j=1}^{i-1} c_j \alpha_j$$

for all sets $\{c_j\}$ of real coefficients where $c_j = \pm 1$ or 0 and where $m = m(n)$ is the number of distinct orthogonality conditions to be satisfied by the element α_n (distinct in the sense that $\alpha_n \perp \beta$ implies $\alpha_n \perp -\beta$ so that conditions obtained one from the other by multiplying the c_j by -1 may be considered as a single condition). Comparison of this statement with the definition of o.p. II will show that the theorem then follows immediately.

The above statement is proved by induction on n . For $n = 2$ it reduces to the statement that in any 2-dimensional Banach space B_2 there are two normal and orthogonal vectors. Let $\alpha_1 \in B_2$ with $\|\alpha_1\| = 1$ be chosen arbitrarily; then consider the functional

$$f(\xi) = \|\xi + \alpha_1\| - \|\xi - \alpha_1\| \quad (\xi \in B_2; \|\xi\| = 1),$$

clearly $f(-\xi) = -f(\xi)$; and since the unit sphere in B_2 is connected and $f(\xi)$ is continuous, it follows that $f(\xi)$ has a zero. This proves the result for $n = 2$.

If the result is true for $n = k - 1$, to prove it for $n = k$ we obviously need only demonstrate the existence in an arbitrary Banach space B of dimensionality $m(k) + 1$ of an element α_k such that $\|\alpha_k\| = 1$ and

$$\alpha_k \perp \sum_{j=1}^{k-1} c_j \alpha_j \quad (c_j = \pm 1 \text{ or } 0; j = 1, 2, \dots, k),$$

where $\{\alpha_j\}$ ($j = 1, 2, \dots, k - 1$) is some set possessing the required orthog-

onality and normality properties. Let us denote the set of sums $\sum_{i=1}^{k-1} c_i \alpha_i$ by $\{\beta_i\}$ ($i = 1, 2, \dots, m(k)$). Then we define the functionals

$$f_i(\xi) = \|\xi + \beta_i\| - \|\xi - \beta_i\| \quad (i = 1, 2, \dots, m(k))$$

for $\xi \in B$ and $\|\xi\| = 1$. For each i , $f_i(\xi)$ is continuous and $f_i(-\xi) = -f_i(\xi)$. Hence, if the unit sphere in B is homeomorphic to the Euclidean $m(k)$ -sphere, it follows ([1; 485, Theorem VIII]) that the f_i have a common zero, and the theorem is proved. To show that the unit sphere in B is homeomorphic to the Euclidean $m(k)$ -sphere we transform B isomorphically into $R_{m(k)+1}$. The image of the unit sphere is a figure which can be mapped by projection along rays from the origin into the Euclidean sphere. Each of these transformations is homeomorphic; hence the combination is.

The following example shows that a construction which proves Theorem 3.9 does not per se prove 3.8. Consider the following vectors in the space $l_\infty^{(5)}$:

$$\alpha = (\frac{3}{4}, \frac{3}{4}, -\frac{3}{4}, 1, 0),$$

$$\beta = (\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, 0, 1),$$

$$\gamma = (0, 1, 1, 0, 0).$$

Here $\|\alpha\| = \|\beta\| = \|\gamma\| = 1$, $\beta \perp \alpha$, $\gamma \perp \alpha$, $\gamma \perp \alpha + \beta$, $\gamma \perp \alpha - \beta$ and $\gamma \perp \beta$. However, β is not $\perp \alpha + \gamma$; furthermore $\|\alpha - \beta + \gamma\| > \|\alpha + \beta + \gamma\|$, and $\|\alpha + \gamma\| > \|\alpha + \beta + \gamma\|$. Thus, while the set $\{\alpha, \beta, \gamma\}$ would suffice in a demonstration of o.p. II, it is of no use in a demonstration of either o.p. I or m.s.p.

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